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Numerical solution for the hyperbolic heat conduction problems in the radial–spherical coordinate system using a hybrid Green's function method

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ABSTRACT

The hyperbolic heat conduction problems in the radial–spherical coordinate system are investigated by the hybrid Green's function method. The present method combines the Laplace transform for the time domain, Green's function for the space domain and ε-algorithm acceleration method for fast convergence of the series solution. Three different examples problems have been analyzed by the present method. It is found that the present method does not exhibit numerical oscillations at the wave front and the numerical solutions are stable.

1. Introduction

Study of the heat conductions problem in cylindrical and spherical coordinate systems has received considerable interest, because of its wide industrial applicability, such as rocket wall, oil reservoirs, and boilers. And for situations involving energy sources such as laser and microwave with extremely short duration or very high frequency and very high temperature gradients, the problems of hyperbolic heat conduction are considered. Researches in the field have been conducted by many investigators [1–18]. The major difficulty encountered in the numerical solutions of hyperbolic heat conduction problems is the suppression of the numerical oscillations in vicinity of sharp discontinuities [11]. However, most the previous works were restricted to analyzing the problem in the rectangular coordinate system. The analytical and numerical solutions of the hyperbolic heat conduction problems in the spherical coordinate system can be found in the literature, J.Y. Lin and H.T. Chen [19], C.S. Tsai, C.Y. Lin and C.I. Hung [20], Fangming jiang and Antonio C. M. Sousa [21] R. Shirmohammadi [22], and R. Shirmohammadi and A. Moosaie [23].

Hence the present study proposes a hybrid Green's function method to analyze the hyperbolic heat conduction problems in the radial–spherical coordinate system. The Laplace transfer method is used to remove the time-dependent terms from the governing equation, and then the s-domain dimensionless temperature function is obtained by the Green's function scheme. Finally, the time-domain dimensionless temperature can be determined by the numerical inversion of the Laplace transform and the ε-algorithm acceleration method.

2. Analysis

Consider the problems of hyperbolic heat conduction in spherical coordinate system. The hyperbolic heat conduction equation is given by

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right]+\frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \phi^2} \quad (1)$$

initial and boundary condition

$$T(\mathbf{r},t)=F(\mathbf{r}),\frac{\partial T}{\partial t}(\mathbf{r},t)=H(\mathbf{r}) \quad \text{for} \quad t=0,$$

$$k_i\frac{\partial T}{\partial n_i}+h_i(T-T_0)=f_i(\mathbf{r},t) \quad \text{on} \quad S_i \quad (2)$$

For convenience of numerical analysis, let us define by the following dimensionless variables

$$\xi=\frac{C^2 t}{2\alpha} \quad (3)$$

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Nomenclature

Bi	Biot number, $\frac{hl}{k}$
C	propagation velocity of thermal wave
c_p	specific heat
f_r	reference heat flux
F	initial condition function
H	initial condition function
G	Green's function
h	thermal convection coefficient
J_ν	Bessel function of the first kind order ν
k	thermal conductivity
n	outward-drawn normal vector to the boundary surface
P_n	Legendre polynomial
r	coordinate
$\mathbf{r}$	space variable
S_i	boundary surface
s	Laplace transform parameter
sum	summation of a series of function
T	temperature
T_0	surrounding temperature
u_k	series

Greek letters

α	thermal diffusivity, $\frac{k}{\rho c_p}$
η	dimensionless length, $\frac{Cr}{2\alpha}$
θ	dimensionless temperature, $\frac{(T-T_0)kc}{\alpha f_r}$
ϕ	coordinate
φ	coordinate
ρ	density
ε	ε-algorithm parameter
μ	coordinate, cos φ
ξ	dimensionless time, $\frac{C^2t}{2\alpha}$

Superscript

—	the Laplace transform
^	dimensionless form

$$\eta = \frac{Cr}{2\alpha} \tag{4}$$

$$\theta = \frac{kC(T-T_0)}{\alpha f_r} \tag{5}$$

The resulting equation becomes

$$\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\theta}{\partial\mu}\right]+\frac{1}{\eta^2(1-\mu^2)}\frac{\partial\theta}{\partial\phi^2} = \frac{\partial^2\theta}{\partial\xi^2}+2\frac{\partial\theta}{\partial\xi} \tag{6}$$

initial and boundary condition

$$\theta(\eta,\mu,\phi,0) = \widehat{F},\ \frac{\partial\theta}{\partial\xi}(\eta,\mu,\phi,0) = \widehat{H},\ \frac{\partial\theta}{\partial\widehat{n}}+\mathrm{Bi}\theta = \widehat{f}_i \ \text{ on } \ S_i. \tag{7}$$

3. Numerical scheme

To remove the ξ-dependent terms, taking the Laplace transform of equation (6) with respect to ξ gives

$$\frac{\partial^2\overline{\theta}}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\overline{\theta}}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\overline{\theta}}{\partial\mu}\right]+\frac{1}{\eta^2(1-\mu^2)}\frac{\partial^2\overline{\theta}}{\partial\phi^2}-(s^2+2s)\overline{\theta} + (s+2)\widehat{F}+\widehat{H}=0 \tag{8}$$

and boundary condition $(\partial\overline{\theta}/\partial\widehat{n})+\mathrm{Bi}\overline{\theta}=\overline{\widehat{f}}_i$ on S_i

To solve the above s-domain heat conduction problem we consider the following auxiliary problem for the same region

$$\frac{\partial^2\overline{G}}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\overline{G}}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2\overline{G}}{\partial\phi^2}+\frac{\partial^2\overline{G}}{\partial\zeta^2}+\delta(\mathbf{r}-\mathbf{r}_0)-(s^2+2s)\overline{G} = 0 \tag{9}$$

$$\frac{\partial\overline{G}}{\partial\widehat{n}}+\mathrm{Bi}\overline{G} = 0, \text{on } S_i$$

The Green's function $\overline{G}(\mathbf{r},s|\mathbf{r}_0,s)$ is determined from equation (9) derived by the method of separated variables [24] and we obtain the s-domain solution $\overline{\theta}(\mathbf{r},s)$ of the heat conduction problem, equation (8) in terms of the Green's function $\overline{G}(\mathbf{r},s|\mathbf{r}_0,s)$ as

$$\overline{\theta}(\mathbf{r},s) = \overline{G}(\mathbf{r},s|\mathbf{r}_0,s)[(s+2)\widehat{F}+\widehat{H}]+\sum_{i=1}^{S}\int_{S_i}\overline{G}(\mathbf{r},s|\mathbf{r}_0,s)|_{r_0=r_i}\overline{\widehat{f}}(\mathbf{r},s)dS_i \tag{10}$$

Where S_i refers to the boundary surface S_i of the region R, $i=1,2,3,\cdots S$ and S in number continuous boundary surfaces.

The dimensionless temperature $\theta(\mathbf{r},\xi)$ can be determined by the numerical inversion of the Laplace transform and the ε-algorithm acceleration method [25]. For this paper, the series solution is taken as 600 terms in a note-book computer.

The ε-algorithm acceleration method, for a non-monotonous $f_N(t) = \sum_{k=1}^{N}u_k$, the$\varepsilon$-algorithm acceleration convergence method is expressed as

Let $N = 2q+1, q\in N$,

$$\mathrm{sum}_m = \sum_{k=1}^{N}u_k \tag{11}$$

and

$$\varepsilon_{p+1}^{(m)} = \varepsilon_{p-1}^{(m+1)}+\frac{1}{\left(\varepsilon_p^{(m+1)}-\varepsilon_p^{(m)}\right)},\ \varepsilon_0^{(m)} = 0,\ \varepsilon_1^{(m)} = \mathrm{sum}_m \tag{12}$$

then the sequence $\varepsilon_1^{(1)}, \varepsilon_3^{(1)}, \varepsilon_5^{(1)}, \ldots, \varepsilon_{2q+1}^{(1)} = \varepsilon_N^{(1)}$, converges to $f_\infty(t)$.

4. Results and discussion

Example 1. One-dimensional region $0 \le \eta \le 1$, a solid sphere problem prescribed wall temperature. The governing equation, the initial and boundary conditions for this case are given by

$$\frac{1}{\eta}\frac{\partial^2}{\partial\eta^2}(\eta\theta) = 2\frac{\partial\theta}{\partial\xi}+\frac{\partial\theta^2}{\partial\xi^2}, \quad 0\le\eta\le 1 \tag{13}$$

$$\theta(\eta,0) = 0, \quad \frac{\partial\theta}{\partial\xi}(\eta,0) = 0 \tag{14}$$

$$\theta(1,\xi) = 1, \tag{15}$$

The Green's function is given by

$$\overline{G}(\eta, s|\eta_0, s) = \frac{2}{\eta\eta_0}\sum_{m=1}^{\infty}\frac{\sin(\beta_m\eta)\sin(\beta_m\eta_0)}{\left(s^2+2s+\beta_m^2\right)} \tag{16}$$

Where β_m's are the positive roots of $\sin(\beta_m) = 0$

The $\theta(\eta, s)$ is obtained as

$$\overline{\theta}(\eta, s) = \frac{2}{\eta}\sum_{m=1}^{\infty}\frac{\beta_m(-1)^{m+1}\sin(\beta_m\eta)}{s\left[\beta_m^2+s^2+2s\right]} \tag{17}$$

Fig. 1 shows the temperature distribution of the one-dimensional a solid sphere problem on hyperbolic heat conduction with a prescribed wall temperature at time $\xi = 0.25, \xi = 0.5$, $\xi = 0.75$ and $\xi = 0.875$. It can be seen that the temperature increases as the time increases because of the wave type heat transfer of the non-Fourier law and the solid sphere geometric structure.

Example 2. One-dimensional region $1 \le \eta \le 2$, hollow sphere problem, the initial and boundary conditions for this case are given by

$$\theta(\eta, 0) = 0, \quad \frac{\partial\theta(\eta, 0)}{\partial\xi} = 0 \tag{18}$$

$$\theta(1, \xi) = 1, \quad \theta(2, \xi) = 0 \tag{19}$$

The Green's function can be obtained

$$\overline{G}(\eta, s|\eta_0, s) = \frac{2}{\eta\eta_0}\sum_{m=1}^{\infty}\frac{\sin[\beta_m(\eta-1)]\sin[\beta_m(\eta_0-1)]}{\left(s^2+2s+\beta_m^2\right)} \tag{20}$$

Where β_m's are the positive roots of $\sin(\beta_m) = 0$.

The $\theta(\eta, s)$ is obtained as

$$\overline{\theta}(\eta, s) = \frac{2}{\eta}\sum_{m=1}^{\infty}\frac{\beta_m\sin[\beta_m(\eta-1)]}{\left(s^3+2s^2+s\beta_m^2\right)} \tag{21}$$

Fig. 2 represents the temperature variation of the one-dimensional hollow sphere problem on hyperbolic heat conduction with a prescribed wall temperature at time $\xi = 0.25, \xi = 0.5$, $\xi = 0.75$ and $\xi = 0.875$. It can be seen that at small time, a wave front advances to the right and the present method solution does not exhibit numerical oscillations at the wave front.

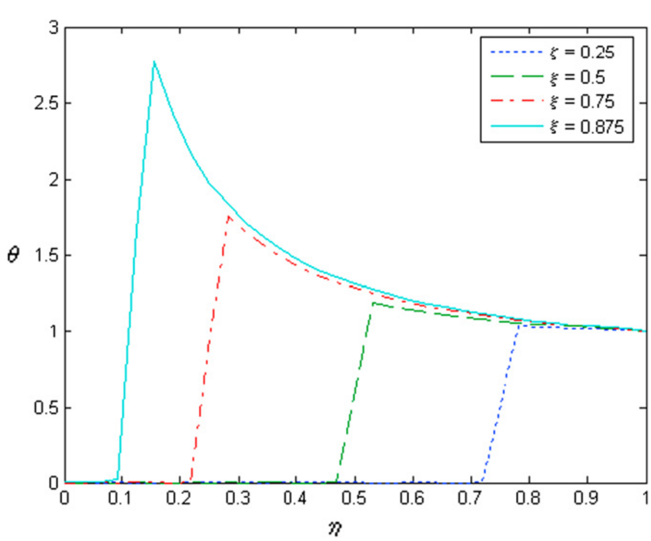

Fig. 1. The temperature distribution of the one-dimensional solid sphere problem for hyperbolic heat conduction with a prescribed wall temperature at time $\xi = 0.25, \xi = 0.5, \xi = 0.75$ and $\xi = 0.875$.

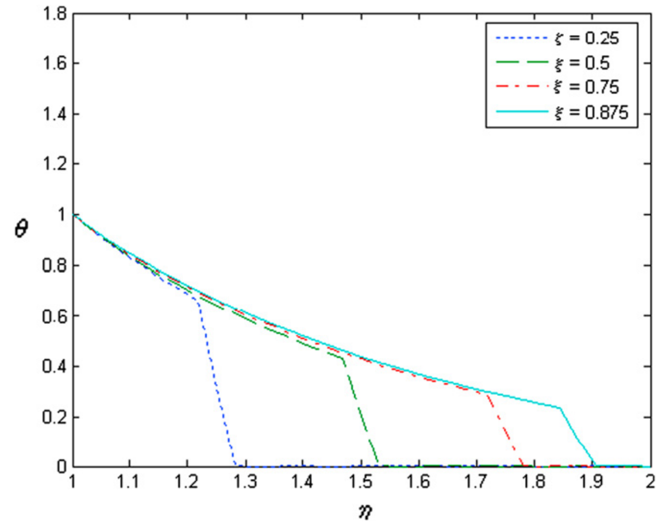

Fig. 2. The temperature variation of the one-dimensional hollow sphere problem on hyperbolic heat conduction with a prescribed wall temperature at time $\xi = 0.25, \xi = 0.5, \xi = 0.75$ and $\xi = 0.875$.

Example 3. One-dimensional region$1 \le \eta \le 2$, hollow sphere problem, the initial and boundary conditions for this case are given by

$$\theta(\eta, 0) = 0, \quad \frac{\partial\theta}{\partial\xi}(\eta, 0) = 0 \tag{22}$$

$$\theta(1, \xi) = 1 \quad \frac{\partial\theta}{\partial\eta}(2, \xi) = 0 \tag{23}$$

The Green's function is given by

$$\overline{G}(\eta, \zeta, s|\eta_0, \zeta_0, s) = \frac{2}{\eta\eta_0}\sum_{m=1}^{\infty}\frac{\sin[\beta_m(\eta-1)]\sin[\beta_m(\eta_0-1)]}{\left(\beta_m^2+s^2+2s\right)} \tag{24}$$

Where β_m's are the positive roots of $\cos(\beta_m) = 0$.

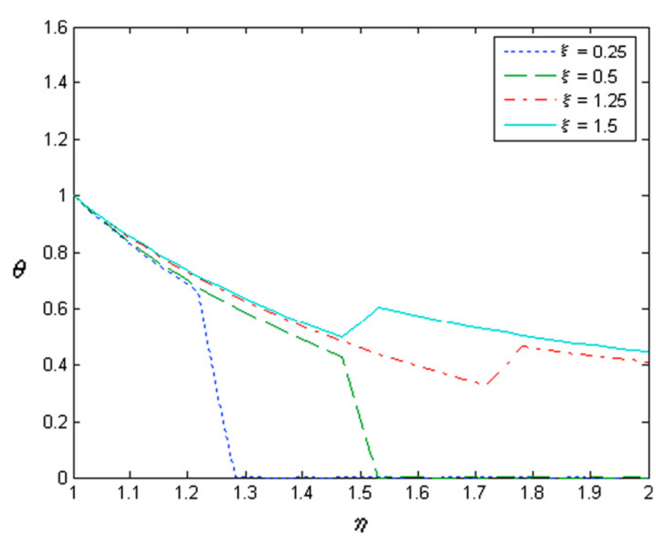

Fig. 3. The one-dimensional region $1 \le \eta \le 2$, hollow sphere problem of dimensionless temperature for (a) $\xi = 0.25$, (b) $\xi = 0.5$, (c) $\xi = 1.25$ and (d) $\xi = 1.5$.

The $\theta(\eta, \zeta, s)$ is obtained as

$$\overline{\theta}(\eta, \zeta, s) = \frac{2}{\eta} \sum_{m=1}^{\infty} \frac{\beta_m \sin[\beta_m(\eta - 1)]}{s\left(\beta_m^2 + s^2 + 2s\right)} \tag{25}$$

Fig. 3 shows the one-dimensional region $1 \leq \eta \leq 2$, hollow sphere problem of dimensionless temperature for time (a) $\xi = 0.25$, (b) $\xi = 0.5$, (c) $\xi = 1.25$ and (d) $\xi = 1.5$. At small time there is a wave front advancing to the right, and at later time, the wave reflected, there is a wave front advancing to the left.

5. Conclusions

The hybrid method has shown success in investigating the hyperbolic heat conduction problem in the radial–spherical coordinate system. To illustrate efficiency of the method, three different examples have been analyzed. It is found from these examples that the present method does not exhibit numerical oscillations at the wave front and the numerical solutions are stable.

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